Eigenvalues and Eignvectors

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Type	Feature	Eigen-values	Eigenvectors	Deduction
Symmetric	$S^T = S = Q\Lambda Q^T$ $= Q\Lambda Q^{-1}$	$\mathbb{R}$	Orthogonal $x_i^T x_j = 0$	$\mathbb{R}$ **eigenvalues**: $Sv = \lambda v \Leftrightarrow \lambda\langle v,v\rangle = \langle \lambda v, v\rangle = \langle Sv, v\rangle = v^T S^T v = v^T \lambda v = \langle v, \lambda v\rangle = \bar{\lambda}\langle v, v\rangle \Longleftrightarrow \boldsymbol{\lambda} = \bar{\boldsymbol{\lambda}} \Longleftrightarrow \boldsymbol{\lambda} \in \mathbb{R}$ **Orthogonal eigenvectors:** $Sx = \lambda_1 x \bigwedge Sy = \lambda_2 y \bigwedge \lambda_1 \neq \lambda_2 \Longleftrightarrow \lambda_1\langle x, y\rangle = \langle \lambda_1 x, y\rangle = x^T S^T y = x^T \lambda_2 y = \lambda_2 \langle x, y\rangle \Leftrightarrow x^T y = 0$
Orthogonal (Unitary)	$Q^T = Q^{-1}(\bar{Q}^T = Q^{-1})$	$\lvert\lambda\rvert = 1$	Orthogonal $\bar{x}_i{}^T x_j = 0$	If S is unitary, then the length of its eigenvalues are 1. Conversely, if S has n orthonormal eigenvectors e_i whose eigenvalues are all of length 1, then S is unitary. If $v = \sum_{i=1}^n c_i e_i$ and $Sv = \lambda v$, then $\lVert v\rVert^2 = \langle v, v\rangle = \sum_{i=1}^n c_i^2$, $\lVert Sv\rVert^2 = \langle Sv, Sv\rangle = \sum_{i=1}^n (c_i\lambda_i)^2 = \sum_{i=1}^n (c_i)^2 = \lVert v\rVert^2$, the length remains after transformation.
Skew-symmetric	$A^T = -A$	$\mathbb{C}\backslash\mathbb{R}$	Orthogonal $\bar{x}_i{}^T x_j = 0$	$\mathbb{C}\backslash\mathbb{R}$ **eigenvalues**: $Sv = \lambda v \Leftrightarrow \lambda\langle v, v\rangle = \langle \lambda v, v\rangle = \langle Sv, v\rangle = v^T S^T v = -v^T \lambda v = \langle v, \lambda v\rangle = -\bar{\lambda}\langle v, v\rangle \Longleftrightarrow \boldsymbol{\lambda} = -\bar{\boldsymbol{\lambda}} \Longleftrightarrow \boldsymbol{\lambda} \in \mathbb{C}\backslash\mathbb{R}$
Complex Hermitian	$\bar{S}^T = S$	$\mathbb{R}$	Orthogonal $\bar{x}_i{}^T x_j = 0$	Complex Symmetric Matrix.
Positive Definite	(i) $pivots > 0$ (ii) $n\ upper\ determinants > 0$ (iii) $\lambda > 0$ (iv) $x^T S x > 0 (x \neq 0)$ (v) $S = A^T A, \det A \neq 0$	$\mathbb{R}^+$	Orthogonal	$S^T = S$ (iv) is the basic definition (energy test) (iv)$\Rightarrow$ (iii) Use Vice Proof, $\exists \lambda \leq 0,\ Sv = \lambda v \Rightarrow v^T S v = \lambda v^T v = \lambda \lVert v\rVert^2 \leq 0!$ and (iii) $\Rightarrow$ (iv) $\lambda_i > 0,\ S = Q\Lambda Q^T; x^T S x = x^T Q \Lambda (x^T Q)^T = x'^T \Lambda x' = \sum_{i=1}^n \lambda_i x_i^2 > 0$ (i)$\Leftrightarrow$(iii) follows from **Sylvester's law of Inertia**. (iii)$\Leftrightarrow$(iv) Mathematical induction (iv)$\Rightarrow$ (v) $S = Q\Lambda Q^T = Q\sqrt{\Lambda}\sqrt{\Lambda}Q^T = Q\sqrt{\Lambda}(Q\sqrt{\Lambda})^T$ (v)$\Rightarrow$ (iv) $S = A^T A,\ x^T S x = x^T A^T A x = \lVert Ax\rVert^2 > 0 (\det A \neq 0)$
Positive Semi-definite	(i) $pivots \geq 0$	$\mathbb{R}^+ \cup \{0\}$	Orthogonal	Notice the restriction is deleted.

	(ii) $n\ upper\ determinants \geq 0$ (iii) $\lambda \geq 0$ (iv) $x^T Sx \geq 0 (x \neq 0)$ (v) $S = A^T A$			
Negative definite	(i) $pivots \leq 0$ (ii) **even upper determinants** $>$ **0, Odd upper determinants < 0** (iii) $\lambda \leq 0$ (iv) $x^T Sx \leq 0 (x \neq 0)$ (v) $S = -A^T A$	$\mathbb{R}^- \cup \{0\}$	Orthogonal	For (ii), A is Negative definite, means $-A$ is positive definite! Minor of $-$ A is greater than 0, then when it comes to even $-$ rank, it's positive for A; when it is odd $-$ rank, it's negative for A! $(-1)^n$
Markov	$m_{ij} > 0, \sum_{i=1}^{n} m_{ij} = 1$	$\lambda_{max} = 1$	Steady state $x > 0$	**Lemma** M^k is also a Markov matrix. The first property is obvious. Then the second property, $u_1 = Mu_0,\ u_0 \geq 0$ *and the sum of* u_0 *is* 1, use column formula, $u_1's$ *sum is also* 1. Then M^k is also a Markov matrix. **Pf** If $\exists |\lambda| > 1$, then M^k will blow up, doesn't fit the property(sum is 1).(10.3)
Similar	$A = BCB^{-1}$	$\lambda(A) = \lambda(C)$ (if and only if)	$Bv(C)$	$Cv = \lambda v \Rightarrow BCv = \lambda Bv \Rightarrow A(Bv) = \lambda(Bv)$ Transitive: $C = XEX^{-1} \Rightarrow A = X'EX'^{-1}$
Projection	$P = P^2 = P^T$	1 0	Column space Null space	Projection on the column space stays the same.
Plane Rotation	$Q = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$	$e^{-i\theta}$ $e^{i\theta}$	$(1, i)$ $(1, -i)$	$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} 1 \\ i \end{pmatrix} = \begin{pmatrix} \cos\theta - i\sin\theta \\ \sin\theta + i\cos\theta \end{pmatrix} = \begin{pmatrix} e^{-i\theta} \\ ie^{-i\theta} \end{pmatrix} = e^{-i\theta}\begin{pmatrix} 1 \\ i \end{pmatrix}$
Reflection	$R = I - 2uu^T (\lVert u \rVert = 1)$	-1 1 $\vdots$ 1	u Whole plane $u^\perp$	$(I - 2uu^T)u = u - 2u = -u$ $v^T u = 0 \Rightarrow (I - 2uu^T)v = v$

Rank One	uv^T	v^Tu 0 $\vdots$ 0	u Whole plane $v^\perp$	$uv^Tu = u(v^Tu) = (v^Tu)u$ $w^Tv = 0 \Rightarrow uv^Tw = u(v^Tw) = 0$
Inverse	A^{-1}	$\frac{1}{\lambda(A)}$	$v(A)$	$Av = \lambda v \Rightarrow \frac{1}{\lambda}v = A^{-1}v$
Shift	$A + cI$	$\lambda(A) + c$	$v(A)$	$Av = \lambda v \bigwedge cIv = cv \Rightarrow (A + cI)v = (\lambda + c)v$
Stable Powers	$A^n \to 0(n \to \infty)$	$\|\lambda\| < 1$	$\forall$	$A = X\Lambda X^{-1} \Rightarrow A^n = X\Lambda^n X^{-1}(\lambda^n \to 0, n \to \infty)$ When not diagonalizable, use Perturbation
Stable Exponential	$e^{At} \to 0(t \to \infty)$	$\mathrm{Re}\,\lambda < 0$	$\forall$	$A = X\Lambda X^{-1} \Rightarrow e^{At} = Xe^{\Lambda t}X^{-1}(\lambda^t \to 0, t \to \infty)$ When not diagonalizable, use Perturbation
Cyclic Permutation	$P_{i,i+1} = 1,\ P_{n1} = 1$ $P_3 = \begin{pmatrix} & 1 & \\ & & 1 \\ 1 & & \end{pmatrix}$	$\lambda_k = e^{\frac{2\pi ik}{n}}$ $=$ roots of 1	$x_k = (1, \lambda_k, \dots, \lambda_k^{n-1})$	**Eigenvalues are roots of 1** $\det(P - \lambda I) = 0 \Longleftrightarrow (-\lambda)^n + (-1)^{n-1} = 0 \Longleftrightarrow \lambda^n = 1$
Circulant	$c_0I + c_1P + \cdots$	λ_k $= c_0$ $+ c_1e^{\frac{2\pi ik}{n}}$	$x_k = (1, \lambda_k, \dots, \lambda_k^{n-1})$	**FFT (Fast Fourier Transform)**
Tridiagonal	$-1,2,-1$ on diagonals	λ_k $= 2$ $- 2\cos\frac{k\pi}{n+1}$	x_k $= \left(\sin\frac{k\pi}{n+1}, \sin\frac{2k\pi}{n+1}, \dots\right)$	By guessing and verifying
Diagonalizable	$A = X\Lambda X^{-1}$	Diagonal of Λ	Columns of X are independent	Diagonalizable condition: $\boldsymbol{A}$ **has** $\boldsymbol{n}$ **independent eigenvectors. (if and only if)** Geometric Multiplicity=Algebraic Multiplicity GM=AM **Lemma** If A has n different eigenvalues, then it has n independent eigenvectors. If not, we can select the <u>minimal</u> linearly dependent eigenvectors. We can assume that $v_k = \sum_{i=1}^{k-1} c_iv_i$, where $c_i \neq 0$. Then $Av_k = \lambda_kv_k = \sum_{i=1}^{k-1}\lambda_kc_iv_i$ and $Av_k = A\left(\sum_{i=1}^{k-1}c_iv_i\right) = \sum_{i=1}^{k-1}\lambda_ic_iv_i$, where $v_i(i = 1,2, \dots, k-1)$ are

				linearly independent. And the same vector has two different forms based on the same basis, which is impossible.
Schur	$A = QTQ^{-1}$, T is triangular	Diagonal of T	Columns of $Q, if\ A^TA = AA^T$	*Schur's Triangularization
Frobenius companion	$D_{i,i+1} = 1,\ D_{nj} = a_{j-1}$	$\lambda_1, \dots, \lambda_n$ are known	$x_k = (1, \lambda_k, \dots, \lambda_k^{n-1})$	Differential Equation $u_{k+n} = a_{n-1}u_{k+n-1} + \cdots + a_1 u_{k+1} + a_0 u_k$ with initial condition $u_0, u_1, \dots, u_{n-1}$ are known. $v_k = D^k v_0 = D^k \begin{pmatrix} u_0 \\ \vdots \\ u_{n-1} \end{pmatrix},\ D^k = X\Lambda^k X^{-1}$ And output u_{k+n-1}. Use the method of standing-by coefficients. And from the diagonalization, we can know that the power of the eigenvalue shares the same value k.
Congruent	$A = BCB^T$, B is invertible	#(positive, zero, negative) index is equal		Sylvester's law of Inertia If and only if positive, negative, zero index for A and C are equal. The proof*
Normal	$\bar{N}^T N = N\bar{N}^T,\ N = Q\Lambda\bar{Q}^T$	$\lvert\lambda\rvert = 1$	Orthonormal vectors in Q	**Proof 1** (i)$N = QTQ^*$, find $T^*T = TT^*$ (ii)$T = \begin{pmatrix} a & b \\ & d \end{pmatrix}$, b=0 (iii)2→ n **Proof 2** $U = TT^*, V = T^*T,\ U(1,1) = V(1,1)$ $U(1,1) = \sum_j T(1,j)T^*(j,1) = \sum_j \lvert T(1,j)\rvert^2$ $V(1,1) = \sum_j T^*(1,j)T(j,1) = \sum_j \lvert T(j,1)\rvert^2 = \lvert T(1,1)\rvert^2$ $\sum_{j=2}^{n} \lvert T(1,j)\rvert^2 = 0 \Leftrightarrow T(1,j) = 0 (j \geq 2)$ The same for other rows. Then T is diagonal.
***Jordan**				

*SVD				

1. A and B are diagonalizable, then $AB = BA \iff A$ and B share all n eigenvectors.
2. $AB = BA \Rightarrow A$ and B share at least 1 eigenvector.(not necessarily diagonalizable)
3. $f(A) = Xf(\Lambda)X^{-1}$
4. (Cayley-Hamilton Theorem) $p_A(A) = 0$